November 8, 2016, 15:00 (Duration: 110 min)

MATH 101/1001 Calculus I Midterm-1

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Department: __________________________________Student Number:____________

In solving the following problems, you are required to show all your work and provide the necessary explanations everywhere to get full credit.

Problem 1. Evaluate the following limits if exists:

a) $\lim\limits_{x\to 1}\frac{\sqrt{x}-x^2}{1-x}$ b) $\lim\limits_{x\to\infty} x(4x-\sqrt{16x^2+2})$

c) $\lim\limits_{x\to 0}\frac{8x}{3\sin x - x}$ d) $\lim\limits_{x\to\infty}\frac{x+\sin x+2\sqrt{x}}{x+\sin x}$

Solution:

a) $$\lim_{x\to 1}\frac{\sqrt{x}-x^2}{1-x}=\lim_{x\to 1}\frac{(\sqrt{x}-x^2)(\sqrt{x}+x^2)}{(1-x)(\sqrt{x}+x^2)}=\lim_{x\to 1}\frac{x-x^4}{(1-x)(\sqrt{x}+x^2)}=\lim_{x\to 1}\frac{x(1-x)(1+x+x^2)}{(1-x)(\sqrt{x}+x^2)}=$$
$$\lim_{x\to 1}\frac{x(1+x+x^2)}{\sqrt{x}+x^2}=\frac{3}{2}$$

Note: It can also be solved by doing a parameter change $\sqrt{x}=u$.

b) $$\lim_{x\to\infty} x(4x-\sqrt{16x^2+2})=\lim_{x\to\infty}\frac{x(4x-\sqrt{16x^2+2})(4x+\sqrt{16x^2+2})}{4x+\sqrt{16x^2+2}}=$$
$$\lim_{x\to\infty}\frac{x(16x^2-16x^2-2)}{4x+\sqrt{16x^2+2}}=\lim_{x\to\infty}\frac{x(-2)}{4x+\sqrt{16x^2+2}}=\lim_{x\to\infty}\frac{-2}{4+\sqrt{16+\frac{2}{x^2}}}=-\frac{1}{4}$$

c)

$$\lim_{x\to 0}\frac{8x}{3\sin x-x}=\lim_{x\to 0}\frac{8}{3\frac{\sin x}{x}-1}=\frac{8}{3(1)-1}=4$$

d)

$$\lim_{x\to\infty}\frac{x+\sin x+2\sqrt{x}}{x+\sin x}=\lim_{x\to\infty}\frac{1+\frac{\sin x}{x}+\frac{2}{\sqrt{x}}}{1+\frac{\sin x}{x}}=\frac{1+0+0}{1+0}=1$$

Problem 2: Explain why the equation $x^3 - 15x + 1 = 0$ has <u>three solutions</u> in the interval [-4, 4] <u>using the Intermediate Value Theorem</u>.

Solution:

Let $f(x) = x^3 - 15x + 1$, which is continuous on $[-4, 4]$. Then $f(-4) = -3$, $f(-1) = 15$, $f(1) = -13$, and $f(4) = 5$. By the Intermediate Value Theorem, $f(x) = 0$ for some x in each of the intervals $-4 < x < -1$, $-1 < x < 1$, and $1 < x < 4$. That is, $x^3 - 15x + 1 = 0$ has three solutions in $[-4, 4]$. Since a polynomial of degree 3 can have at most 3 solutions, these are the only solutions.

Problem 3: Evaluate the derivatives (dy/dx) of the given functions.

a. $y = \cos^3 \left(\frac{x}{x+1}\right)$

b. $y = \sin(\cos(3x - 1))$

Solution:

Problem 4: Let $f(x) = x/(x+1)$. Use the definition of the derivative to find $\boldsymbol{f'}(x)$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x+h)(x+1)}{(x+h+1)(x+1)} - \frac{(x+h+1)x}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x+h)(x+1) - (x+h+1)x}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x^2 + x + hx + h) - (x^2 + hx + x)}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x^2 + x + hx + h - x^2 - hx - x}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{(x+h+1)(x+1)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h}{(x+h+1)(x+1)} \cdot (x+h+1)(x+1)}{h \cdot (x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \frac{h}{h \cdot (x+h+1)(x+1)}$$

$$= \lim_{h \to 0} \frac{1}{(x+h+1)(x+1)}$$

$$= \frac{1}{(x+0+1)(x+1)}$$

$$= \frac{1}{(x+1)(x+1)}$$

$$= \frac{1}{(x+1)^2}$$

Therefore $f'(x) = \frac{1}{(x+1)^2}$.

Problem 5: The sides a and b of a right triangle are changing at the rates of $\frac{da}{dt} = -3t\ m/sec$ m/sec and $\frac{db}{dt} = 2t\ m/sec$ respectively. How fast does the area of the triangle change when $a = 30$ m and $b = 40$ m at $t = 2$ sec?

Solution: The area of a triangle is given by $A = \frac{1}{2}ab$.

Then $\frac{dA}{dt} = \frac{1}{2}(\frac{da}{dt}b + a\frac{db}{dt})$. So,

$$\frac{dA}{dt} = \frac{1}{2}(-6.40 + 30.4) = -60\ \ m^2/sec\ .$$

Problem 6: a) Write down a function which is continuous at x=1 but not differentiable at x=1. **b)** Write down a function $\lim_{x\to 1-} f(x) = f(1)$ but f is not continuous at $x = 1$.

Solution:

a) $f(x) = |x - 1|$

b) $\text{f(x)} = \begin{cases} x, & if\ x \le 1 \\ 0, & if\ x > 1 \end{cases}$

Problem 7: Find the lines that are (a) tangent and (b) normal to the following curve at the point (1, π/2): $\boldsymbol{2xy + \pi \sin y = 2\pi}$

Solution:

$2xy + \pi \sin y = 2\pi \Rightarrow 2xy' + 2y + \pi(\cos y)y' = 0 \Rightarrow y'(2x + \pi\cos y) = -2y \Rightarrow y' = \frac{-2y}{2x+\pi\cos y}$;

(a) the slope of the tangent line $m = y'\Big|_{(1,\frac{\pi}{2})} = \frac{-2y}{2x+\pi\cos y}\Big|_{(1,\frac{\pi}{2})} = -\frac{\pi}{2} \Rightarrow$ the tangent line is $y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1)$

$\Rightarrow y = -\frac{\pi}{2}x + \pi$

(b) the normal line is $y - \frac{\pi}{2} = \frac{2}{\pi}(x-1) \Rightarrow y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$

Problem 8: Consider the graph of $x^2 + y^2 = 49$.

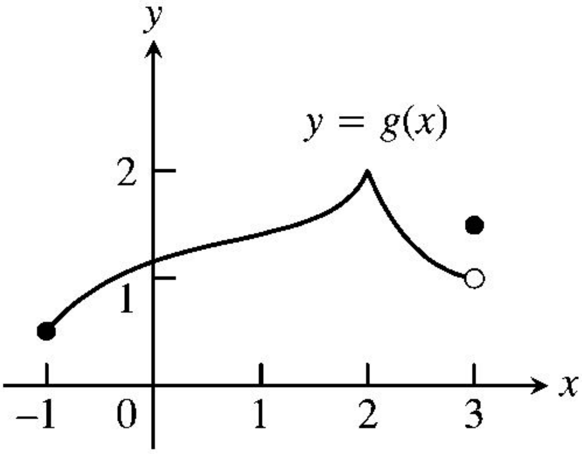
a) What would be the new equation if the graph is shifted 3 units down and 2 units left?

b) Sketch the graph of the new equation.

Solution: To make a vertical shift of 3 units down add 3 units to y value. To make a horizontal shift of 2 units left add 2 units to x value. New equation: $(x + 2)^2 + (y + 3)^2 = 49$.

The original graph is a circle centered at (0,0) with radius 7. The new graph is the same circle whose center is at (-2,-3).

Problem 9: Below is the graph of a function g(x) defined in the interval [1, 3]:



a) At which points g(x) does not have limit?
b) At which points g(x) is not continuous?
c) At which points g(x) is not differentiable?

Solution:
a) None
b) x=3
c) x=2 and x=3